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On determination of the characteristic equations for the stability of parallel flows

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The Orr–Sommerfeld equation is solved for large Reynolds number by use of inner and outer expansion theory. The method is shown to have distinct computational advantages over the method of solution due to Sommerfeld and Lin and to be applicable to a wider class of boundary conditions. The method can be simply extended to other characteristic problems in fluids involving viscous effects concentrated in narrow regions.

1. Introduction

The mathematical formulation of the problem of the stability of parallel flows of a viscous fluid is due to Orr (1906–7) and Sommerfeld (1908), the first solution by Heisenberg (1924) being corrected and clarified by Lin (1945*a*). The original boundary-value problems dealt with considered boundaries which were solid walls, or else considered flows which eventually become uniform (i.e. boundary-layer flows). Many recent authors have used the same techniques for boundaries which were flexible or interfacial. The question of the validity of Heisenberg's technique could be raised in these cases, since it is anticipated that such boundaries would involve viscous layers of thicknesses not contemplated in Heisenberg's original solutions. Accordingly, we consider here the use of inner and outer expansion techniques† as an alternative approach to the problem. It is shown that this method involves a considerable reduction in algebra and analysis in that coefficients and eigenvalues are determined step by step in an automatic manner rather than through solutions of determinants, and also contributes to a clearer understanding of the action of viscosity in all regions as well as the nature and profile of the disturbance. The method also can be extended conveniently to problems involving thermal effects, non-Newtonian effects, etc., and provides for higher-order corrections.‡

It is convenient first to summarize the previous approach to the problem. For a parallel flow with velocity $U(y)$, a small wavy perturbation in the velocity results in the Orr–Sommerfeld equation

$$(1/\alpha R)(D^2-\alpha^2)\,v = iW(D^2-\alpha^2)\,v - ivD^2W, \tag{1}$$

where $W = U-c$, c is the (complex) wave speed, α the wave-number of the disturbance, R a Reynolds number, D denotes differentiation with respect to y, and

† See, for instance, Van Dyke (1964).

‡ The approach of Duty & Reid (1964) for the stability of Couette flow at high shear rates is equivalent to the present method to the lowest order.

v represents the y-dependent part of the vertical component of the disturbance velocity. At a rigid boundary, v and Dv (which is proportional to the horizontal components of the velocity) must both vanish. From consideration of the inviscid

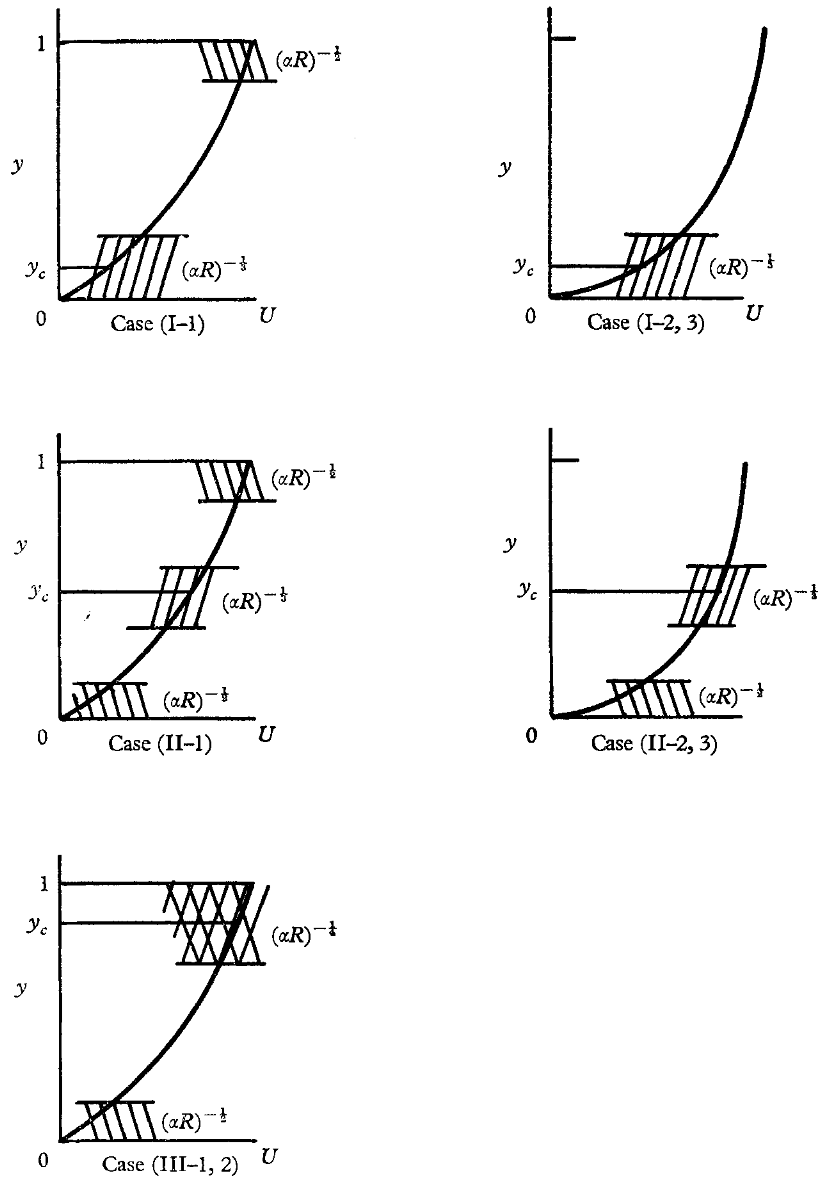

FIGURE 1. Location and order of thickness of the inner regions.

limit (see Lin 1945*b*, or Lin 1955, p. 55ff.) for undamped waves (c real), W is expected to be zero somewhere in the flow régime, say at $y = y_c$.

The expectation that αR is large for the onset of instability suggests an asymptotic solution to equation (1). The approach of Heisenberg and Lin to the problem in essence compared the solution of equation (1) with that of an equation with constant coefficients by taking

$$v \sim e^{(\alpha R)^{\frac{1}{2}} Q(y)} [f_0(y) + (\alpha R)^{-\frac{1}{2}} f_1(y) + \ldots]. \tag{2}$$

The highest-order terms obtained by substitution of this into equation (1) determine four possible values of Q; they are found to be a double root $Q = 0$, giving rise to two inviscid solutions, and $Q = \pm \int (iW)^{\frac{1}{2}} dy$. The branch point in the integral at y_c (the Stokes, critical, or turning point) raises the question of the analytic continuation of the functions across y_c. This was first satisfactorily accomplished by Lin, who used comparison with an equation with linear coefficients (the so-called WKBJ method). Here the stretched co-ordinate $\eta = (\alpha R)^{-\frac{1}{3}} (y - y_c)$ is introduced, the approximation in this case giving the four solutions 1, η, $\chi_3(\eta)$, $\chi_4(\eta)$, where

$$\chi_3(\eta) = \int_\infty^\eta d\eta \int_\infty^\eta d\eta\, \eta^{\frac{1}{2}} H_{\frac{1}{3}}^{(1)} [\tfrac{2}{3} (DW_c)^{\frac{1}{2}} (i\eta)^{\frac{3}{2}}], \tag{3}$$

$$\chi_4(\eta) = \int_\infty^\eta d\eta \int_\infty^\eta d\eta\, \eta^{\frac{1}{2}} H_{\frac{1}{3}}^{(2)} [\tfrac{2}{3} (DW_c)^{\frac{1}{2}} (i\eta)^{\frac{3}{2}}], \tag{4}$$

the subscript c indicating evaluation at y_c. The two inviscid solutions are used in conjunction with either χ_3 and χ_4, or the values given by equation (2) with the remaining roots of Q giving, through use of the boundary conditions, four homogeneous algebraic equations, hence a four by four determinant to evaluate in order to determine c as a function of R and α.

The approach just described attempts to find an asymptotic solution valid throughout the flow region. It fails at the critical point, and the WKBJ is used then to patch the solution at this point. The method adopted in the present approach is to find solutions valid in various pieces of the flow region, and then to join these piecewise solutions to one another in intermediate regions. While the approaches have much in common, the present method is believed to be more straightforward and to provide more simply an error estimate, as well as providing an algorithm for obtaining higher approximations (Lin 1955, p. 129).

The use of any asymptotic method in the solution of equation (1) requires some knowledge of the magnitude of c. Accordingly, we divide the problem into three cases depending on the location of the critical point. These are sufficient to demonstrate the method. The cases are illustrated schematically in figure 1.

2. Case I. Critical point near a rigid stationary boundary

In the solution of equation (1) for large αR, one anticipates that throughout most of the flow region (an 'outer' region) the derivatives of v with respect to y will all be of the same order. In this case the right side of the equation is set to zero, giving the inviscid form of the stability equation. However, near the boundaries, viscous effects must be present in the form of a boundary layer in order to satisfy the boundary conditions. In such a region (an inner region), it is well known that co-ordinate stretching is necessary to provide a 'boundary-layer' equation, the solutions of which are then to be matched to the inviscid solutions. In general, equation (1) will also have an interior shear layer at y_c;

for small c this layer is at the boundary rather than in the interior and in fact dictates the thickness of the layer.

We take the lower boundary to be a rigid wall at $y = 0$. Letting $y - y_c = \mu\eta$, where $\mu = O(\alpha R)^{-n}$, it is known (Lin 1945*a*) that $\mu = (\alpha R)^{-\frac{1}{3}}$ is the proper choice for n; this then implies that $y_c \leqslant O(\mu)$ for this scaling. The asymptotic form used in equation (2) anticipates that v is expressible as a Taylor series in inverse powers of the Reynolds number in all regions of the flow. In light of more recent techniques in singular perturbation theory we let

$$v^*(y,\mu) \sim \sum_{n=0}^{\infty} \epsilon_n^*(\mu)\, v_n^*(\eta), \tag{5}$$

in the steep inner region, and

$$\bar{v}(y,\mu) \sim \sum_{n=0}^{\infty} \bar{\epsilon}_n(\mu)\, \bar{v}_n(y), \tag{6}$$

in the outer region, where *a priori* we specify only that

$$\lim_{\mu\to 0} \epsilon_{n+1}^*/\epsilon_n^* = \lim_{\mu\to 0} \bar{\epsilon}_{n+1}/\bar{\epsilon}_n = 0,$$

and allow the ϵ_n^* and $\bar{\epsilon}_n$ to be determined by the matching procedure. However, for a problem which is both homogeneous and linear,† and in which μ appears in the differential equation only in polynomial form, one set (say ϵ_n^*) has a high degree of arbitrariness. We take $\epsilon_n^* = \mu^n$ in the present case as a suitable choice. In the case under consideration ($c \leqslant O(\mu)$), it is in fact usually sufficient to utilize only the ϵ_0^*, $\bar{\epsilon}_0$ and $\bar{\epsilon}_1$ terms in the expansion for a first approximation to c.

Substituting equation (5) into equation (1), the equation obtained is

$$\begin{aligned} 0 = \frac{d^4v^*}{d\eta^4} - i\eta DW_c \frac{d^2v^*}{d\eta^2} + \mu i D^2W_c\left[-\tfrac{1}{2}\eta^2\frac{d^2v^*}{d\eta^2} + v^*\right] \\ + \mu^2\left[-\left(\frac{i}{6}\eta^3D^3W_c + 2\alpha^2\right)\frac{d^2v^*}{d\eta^2} + i\eta(\alpha^2DW_c + D^3W_c)\,v^*\right] \\ + \text{ terms involving } \mu \text{ to the third and higher powers.} \end{aligned}$$

To the lowest order this is the equation used by Lin at the critical point; thus v^* is a linear combination of $1, \eta, \chi_3$, and χ_4. But χ_4 is not of boundary-layer type; it grows exponentially as $\eta \to +\infty$ and hence cannot be matched to the inner solution. The combination of the other three solutions satisfying $v_0^* = Dv_0^* = 0$ at $y = 0$ is then

$$v_0^* = -\eta_0[1 - F(\eta_0)] + \eta - \chi_3(\eta)/\chi_3'(\eta_0), \tag{7}$$

where $\eta_0 = -y_c/\mu$ and $F(\eta_0) = \chi_3(\eta_0)/\eta_0\chi_3'(\eta_0)$.

For the outer region, substitution of equation (6) into equation (1) gives

$$WD^2\bar{v}_n - (D^2W + \alpha^2W)\,\bar{v}_n = 0, \tag{8}$$

for n such that $\bar{\epsilon}_n > O(\mu^3\bar{\epsilon}_0)$. Heisenberg solved equation (8) by expansion in powers of α^2. (This is equivalent to transforming equation (8) into an integral

† If it were decided that ϵ_N^*, say, was to be of the form $\ln\mu$ (or any other transcendental function of μ), then $v_N^* = v_0^*$, $\epsilon_{N+1}^* = \epsilon_1^* \ln\mu$, $v_{N+1}^* = v_1^*$, etc. The effect then is the same as multiplying v^* (and $\bar{v}$) by $(1+\ln\mu)$. Similarly, when computing v_n^*, $n \geqq 1$, only the exponentially decaying and constant parts of the homogeneous solution should be used.

equation and using the method of iterated kernels.) The solutions found in this manner are

$$\Phi_1(y,y_1) = W(y)\sum_{n=0}^{\infty}\alpha^{2n}h_n(y,y_1), \tag{9}$$

$$\Phi_2(y,y_1) = W(y)\sum_{n=0}^{\infty}\alpha^{2n}k_n(y,y_1), \tag{10}$$

where

$$h_0 = 1, \quad h_n = \int_{y_1}^{y} dy\, W^{-2}\int_{y_1}^{y} dy\, W^2 h_{n-1},$$

$$k_0 = \int_{y_1}^{y} dy\, W^{-2}, \quad k_n = \int_{y_1}^{y} dy\, W^{-2}\int_{y_1}^{y} dy\, W^2 k_{n-1}.$$

Here y_1 is an arbitrary reference point. For matching purposes, it is convenient to express these solutions also in power series. Letting $z = y - y_c$, an alternative pair of solutions is easily found to be

$$\phi_1(z) = z + \frac{1}{2}\left(\frac{D^2W_c}{DW_c}\right)z^2 + \frac{1}{6}\left[\alpha^2 + \frac{D^3W_c}{DW_c}\right]z^3 + \ldots, \tag{11}$$

$$\phi_2(z) = \phi_1(z)\ln z + \frac{DW_c}{D^2W_c} + \frac{1}{2}\left[\frac{DW_c}{D^2W_c}\alpha^2 + \frac{D^3W_c}{D^2W_c} - \frac{3}{2}\frac{D^2W_c}{DW_c}\right]z^2 + \ldots \tag{12}$$

(these correspond to Tollmien's (1935) solutions), where a particular choice of y_1 gives

$$\left.\begin{aligned} \phi_i(z) &= \beta_i\Phi_1(y,0) + \gamma_i\Phi_2(y,0), \\ \beta_i = \phi_i(-y_c)/W_0, &\quad \gamma_i = W_0(d\phi_i/dz)_{z=-y_c} - \phi_i(-y_c)\,DW_0. \end{aligned}\right\} \tag{13a}$$

For $W_0 = -c$, from equations (11) and (12),

$$\left.\begin{aligned} \beta_1 &\approx -y_c/W_0, & \gamma_1 &\approx -\tfrac{1}{2}y_cW_0DW_c/D^2W_c, \\ \beta_2 &\approx DW_c/W_0D^2W_c, & \gamma_2 &\approx -DW_cDW_0/D^2W_c. \end{aligned}\right\} \tag{13b}$$

To match $\bar{v}$ to v^* it is convenient to use ϕ_1 and ϕ_2, replacing z by $\mu\eta$. In matching $\phi_1(\mu\eta)$ and $\phi_2(\mu\eta)$ to v_0^*, it appears that a proper choice is $\bar{\epsilon}_0 = \mu^{-1}, \bar{v}_0 = \phi_1$ to merge with the linear term in v_0^*, and

$$\bar{\epsilon}_1 = 1, \quad \bar{v}_1 = -\eta_0[1-F(\eta_0)]\,(D^2W_c/DW_c)\,\phi_2 + A\phi_1$$

to merge with the constant. Since the merging is done at large η and since $\lim_{\eta\to\infty}\chi_3 \leqslant O(e^{-\eta})$, no terms are needed to merge with χ_3. The constant A (of order unity) can only be determined after $v_1^*(\eta)$ is known. The appearance of ϕ_2 in $\bar{v}_1$, however, introduces a $\mu\ln\mu$ term which cannot be matched with the inner solution nor balanced by higher approximations. This must instead be balanced by a lower-order term in the expansion. The selection just indicated is in fact valid for $\bar{\epsilon}_2$ and $\bar{v}_2$; for $\bar{\epsilon}_1$ we must use $\ln\mu$, and then

$$\bar{v}_1 = \eta_0[1-F(\eta_0)]\,(D^2W_c/DW_c)\,\phi_1.$$

Thus in the inviscid region

$$\begin{aligned}\bar{v}(y) \sim (1/\mu)\,\phi_1(z) - \eta_0[1-F(\eta_0)]\,(D^2W_c/DW_c)& \\ \times(\phi_2(z) - \phi_1(z)\ln\mu) + A\phi_1(z) + O(\mu\ln\mu).&\end{aligned} \tag{14}$$

Equation (14) is sufficient to determine the characteristic equation to the lowest non-trivial order for this order of magnitude of y_c for the cases considered by Lin.

Case (I–1). *Moving boundary at* $y = 1$

To satisfy the no-slip conditions at $y = 1$, a boundary layer of order $(\alpha R)^{-\frac{1}{2}}$ must exist at $y = 1$ in which the solutions will be of the type indicated in the first inner region of Case II. Letting $\xi = -(1-y)(\alpha R)^{\frac{1}{2}}$, equation (14) becomes

$$\bar{v} \sim \frac{1}{\mu}\left\{(\phi_1) + \mu\left[-\eta_0(1-F(\eta_0))\frac{D^2W_c}{DW_c}(\phi_2)_1[(\phi_2(z))_1 - (\phi_1(z))_1 \ln\mu] + A(\phi_1)_1\right]\right.$$
$$\left. - \mu^{\frac{3}{2}}\xi\left(\frac{d\phi_1}{dz}\right)_1 + O(\mu^2\ln\mu)\right\},$$

which merges with

$$v^{**} \sim 1/\mu\left\{(\phi_1)_1 + H\,e^{\xi(iW_1)^{\frac{1}{2}}} + \mu\ln\mu[(\phi_1)_1\eta_0(1-F(\eta_0)) + K\,e^{\xi(iW_1)^{\frac{1}{2}}}]\right.$$
$$+\mu[\eta_0(F(\eta_0)-1)\,D^2W_c/DW_c\,(\phi_2)_1 + A(\phi_1)_1 + J\,e^{\xi(iW_1)^{\frac{1}{2}}}]$$
$$\left.+\mu^{\frac{3}{2}}\left[-\left(\frac{d\phi_1}{dz}\right)_1\xi + e^{\xi(iW_1)^{\frac{1}{2}}}\left(M + \tfrac{1}{2}i\frac{DW_1}{(iW_1)^{\frac{1}{2}}}H\xi\right)\right] + O(\mu^2\ln\mu)\right\}$$

in the layer of thickness $(\alpha R)^{-\frac{1}{2}}$. The subscript 1 indicates evaluation *at* $y = 1$. Imposing the boundary conditions $v^{**} = dv^{**}/d\xi = 0$ at $\xi = 0$, the characteristic equation is

$$\left(\frac{\phi_1}{\phi_2}\right)_1 = \frac{\mu\left[\eta_0(1-F(\eta_0))\dfrac{D^2W_c}{DW_c}\right] + \left(\dfrac{d\phi_1}{dz}\Big/\phi_2\right)_1\left(\dfrac{\mu^{\frac{3}{2}}}{(iW_1)^{\frac{1}{2}}}\right)}{1 + \eta_0[1-F(\eta_0)]\,\mu\ln\mu + A\mu + (DW_1/2W_1)\,\mu^{\frac{3}{2}}}.$$

Case (I–2*a*). *Antisymmetrical disturbance in a flow symmetrical about* $y = 1$.

The boundary conditions are $v = D^2v = 0$ at $y = 1$. Since $\bar{v}$ satisfies equation (8), only one of these need be applied. Then from equation (14),

$$\left(\frac{\phi_1}{\phi_2}\right)_1 \approx \frac{\eta_0\mu[1-F(\eta_0)]}{(DW_c/D^2W_c) + \eta_0\mu[1-F(\eta_0)]\ln\mu}.$$

Using equations (13), this is equivalent in the lowest order to

$$\frac{W_0\,\Phi_2(1,0)}{\Phi_2(1,0)\,DW_0 - W_0^{-1}\,\Phi_1(1,0)} \approx \frac{F(\eta_0)}{(DW_0/W_0) + \frac{1}{2}(DW_c/D^2W_c)} \approx -y_cF(\eta_0),$$

in accordance with Lin (1945*a*), equation (6.15). (Note his mislabelling of the symmetry.)

Case (I–2*b*). *Symmetrical disturbance in a flow symmetrical about* $y = 1$

In this case odd derivatives of W and v must vanish at $y = 1$, and hence from equation (14)

$$\left(\frac{D\phi_1}{D\phi_2}\right)_1 \approx \frac{\eta_0\mu[1-F(\eta_0)]}{(DW_c/D^2W_c) + \eta_0\mu[1-F(\eta_0)]\ln\mu},$$

or
$$\frac{W_0\,D\Phi_2(1,0)}{DW_0\,D\Phi_2(1,0) - W_0^{-1}\,D\Phi_1(1,0)} \approx \frac{F(\eta_0)}{DW_0/W_0 + \frac{1}{2}(DW_c/D^2W_c)} \approx -y_cF(\eta)_0,$$

in accordance with Lin (1945*a*), equation (6.14).

Case (I–3). *Flow of boundary layer type*

Here $\bar{v} = O(e^{-\alpha y})$ as $y \to \infty$; this implies $(D\bar{v}+\alpha\bar{v})_{y\to\infty} = 0$, giving

$$\left(\frac{D\phi_1+\alpha\phi_1}{D\phi_2+\alpha\phi_2}\right)_{z=\infty} \approx \frac{\eta_0\mu[1-F(\eta_0)]}{(DW_c/D^2W_c)+\eta_0\mu[1-F(\eta_0)]\ln\mu},$$

or

$$\frac{D\Phi_2(\infty,0)+\alpha\Phi_2(\infty,0)}{DW_c[D\Phi_2(\infty.0)+\alpha\Phi_2(\infty,0)]-W_0^{-1}[D\Phi_1(\infty,0)+\alpha\Phi_1(\infty,0)]}$$

$$\approx \frac{F(\eta_0)}{(DW_0/W_0)+\frac{1}{2}(DW_c/D^2W_c)} \approx -y_cF(\eta_0),$$

in accordance with Lin (1945*a*), equation (6.17).

Note that the determination of the argument of z in the logarithmic term was accomplished in this process by the matching of $\bar{v}$ to v^*. The necessity of going around the critical point in carrying out the integrals in the Φ solutions can be avoided completely by working with $\Phi(y, 2y_c)$, say, instead of $\Phi(y,0)$. The use of the latter in this case was solely for comparison with Lin's work.

Another interesting feature of the present solution is that the matching procedure dictates that for small c some of the $\bar{\epsilon}_n$ must be of the form $\mu^m \ln\mu$, which is not anticipated by equation (2). These logarithmic terms are forced by the particular solutions in the higher-order approximations, and are now well known in boundary-layer theory as well as in low-Reynolds-number flows. While these are unimportant in the lowest approximation, they become vital in continuing the solution to higher approximations.

3. Case II. Critical point in the interior of the flow away from boundaries and extremum values of W

For c no longer small, the shear layer with thickness of order $\mu = (\alpha R)^{-\frac{1}{3}}$ moves to the interior of the flow, and a new layer with thickness of order $\epsilon = (\alpha R)^{-\frac{1}{2}}$ takes its place at the rigid boundary. Thus at least four expansions are in general necessary:

$$v^*(\zeta,\epsilon) \sim \sum_{n=0}^{\infty}\epsilon_n^*(\epsilon)\,v_n^*(\zeta) \quad \text{for} \quad 0 \leqslant \zeta = y/\epsilon = O(1) \quad \text{(inner region 1)}, \tag{15}$$

$$\bar{v}(y,\epsilon) \sim \sum_{n=0}^{\infty}\bar{\epsilon}_n(\epsilon)\,\bar{v}_n(y) \quad \text{for} \quad 0 < y < y_c \quad \text{(outer region 1)}, \tag{16}$$

$$v^{**}(\eta,\mu) \sim \sum_{n=0}^{\infty}\epsilon_n^{**}(\mu)\,v_n^{**}(\eta) \quad \text{for} \quad \eta = (y-y_c)/\mu = O(1) \quad \text{(inner region 2)} \tag{17}$$

and
$$\bar{\bar{v}}(y,\mu) \sim \sum_{n=0}^{\infty}\bar{\bar{\epsilon}}_n(\mu)\,\bar{\bar{v}}_n(y) \quad \text{for} \quad y > y_c \quad \text{(outer region 2)}. \tag{18}$$

Taking $y = 0$ to be a rigid boundary as in the previous case, again for convenience we set $\epsilon_n^* = \epsilon^n$. Rewriting equation (1) in terms of ζ,

$$0 = \frac{d^4v^*}{d\zeta^4} - iW_0\frac{d^2v^*}{d\zeta^2} - \epsilon iDW_0\zeta\frac{d^2v^*}{d\zeta^2} + \epsilon^2\left[-2\alpha^2\frac{d^2v^*}{d\zeta^2} + iW_0\alpha^2v^* + D^2W_0 \right.$$
$$\left. \times\left(-\tfrac{1}{2}i\zeta^2\frac{d^2v^*}{d\zeta^2} + iv^*\right)\right] + \text{terms involving } \epsilon \text{ to the third and higher powers,}$$

where a zero subscript indicates evaluation at $y = 0$. The first solution satisfying $v^* = Dv^* = 0$ at $\zeta = 0$ as well as being of boundary-layer type (i.e. it does not grow exponentially) is

$$v_0^* = 1 - \zeta(iW_0)^{\frac{1}{2}} - e^{-\zeta(iW_0)^{\frac{1}{2}}}, \tag{19}$$

$\sqrt{i}$ being the value in the first quadrant. Higher approximations yield

$$v_1^* = \frac{DW_0}{(iW_0^3)^{\frac{1}{2}}} [(\tfrac{5}{4} + \tfrac{5}{4}\zeta(iW_0)^{\frac{1}{2}} + \tfrac{1}{4}iW_0\zeta^2)\, e^{-\zeta(iW_0)^{\frac{1}{2}}} - \tfrac{5}{4}] \tag{20}$$

and

$$\begin{aligned} v_2^* = \frac{1}{6}\left(\frac{D^2W_0}{W_0} + \alpha^2\right) (3\zeta^2 - \zeta^3 (iW_0)^{\frac{1}{2}}) + k(1 - e^{-\zeta(iW_0)^{\frac{1}{2}}}) \\ + \left[-\left(\frac{DW_0}{W_0}\right)^2 \left(\tfrac{151}{32}\zeta(iW_0)^{-\frac{1}{2}} + \tfrac{55}{32}\zeta^2 + \tfrac{17}{48}\zeta^3(iW_0)^{\frac{1}{2}}\frac{iW_0}{32}\zeta^4\right) \right. \\ \left. + \frac{D^2W_0}{W_0}\tfrac{13}{8}\zeta(iW_0)^{-\frac{1}{2}} + \tfrac{5}{8}\zeta^2 + 12\,\zeta^3 (iW_0)^{\frac{1}{2}} \right] e^{-\zeta(iW_0)^{\frac{1}{2}}}, \end{aligned} \tag{21}$$

where
$$k = \frac{1}{iW_0}\left[-\frac{13}{8}\frac{D^2W_0}{W_0} + \frac{151}{32}\left(\frac{DW_0}{W_0}\right)^2 - \frac{\alpha^2}{2}\right].$$

For $\bar{v}$, again equation (8) will be used for the first approximations. Near $y = 0$,

$$\Phi_1(y,0) = W_0 + yDW_0 + \tfrac{1}{2}(\alpha^2W_0 + D^2W_0)\, y^2 + \tfrac{1}{6}(D^3W_0 + \alpha^2DW_0)\, y^3 + \ldots,$$

$$\Phi_2(y,0) = \frac{1}{W_0}\left[y + \frac{1}{6}\left(\frac{D^2W_0}{W_0} + \alpha^2\right) y^3 + \frac{1}{12}\left(\frac{D^3W_0}{W_0} - \frac{DW_0D^2W_0}{W_0^2}\right) y^4 + \ldots\right],$$

so, to match v_0^* and v_1^*,

$$\bar{\epsilon}_0 = 1/\epsilon, \quad \bar{v}_0 = -W_0(iW_0)^{\frac{1}{2}}\,\Phi_2(y,0),$$

$$\bar{\epsilon}_1 = 1, \quad \bar{v}_1 = (1/W_0)\,\Phi_1(y,0) - DW_0\,\Phi_2(y,0).$$

To match further, $\bar{\epsilon}_2 = \epsilon$, and so $\bar{v}_2$ must satisfy

$$iW(D^2 - \alpha^2)\,\bar{v}_2 - i\bar{v}_2 D^2W = (D^2 - \alpha^2)^2\,\bar{v}_0.$$

Ordinarily $\bar{v}_2$ is not needed except for higher approximations to the characteristic equation.

At the interior shear region, neither χ_3 nor χ_4 are of boundary-layer type, so they cannot be used for matching. Picking a $y_2 < y_c$ (corresponding to $z_2 = y_2 - y_c$) and of convenient size for computation, and letting

$$\Phi_i(y,0) = m_i\,\phi_1(z) + n_i\,\phi_2(z)$$

(the $\Phi_0^i(y,0)$ are convenient for this case since they are regular near $y = 0$), where

$$m_i = \frac{\Phi_i(y_2,0)\, D\phi_2(z_2) - \phi_2(z_2)\, D\Phi_i(y_2,0)}{\phi_1(z_2)\, D\phi_2(z_2) - \phi_2(z_2)\, D\phi_1(z_2)},$$

$$n_i = \frac{\phi_1(z_2)\, D\Phi_i(y_2,0) - \Phi_i(y_2,0)\, D\phi_1(z_2)}{\phi_1(z_2)\, D\phi_2(z_2) - \phi_2(z_2)\, D\phi_1(z_2)},$$

$\bar{v}$ can be expressed in the inner variables as

$$\begin{aligned} \bar{v} \sim \mu^{-\frac{3}{2}}[in_2(iW_0)^{\frac{3}{2}}\,(DW_c/D^2W_c)] + \mu^{-\frac{1}{2}}\ln\mu[im_2(iW_0)^{\frac{3}{2}}\,\eta] + \mu^{-\frac{1}{2}}i(iW_0)^{\frac{3}{2}}\,[m_2\eta + n_2\eta\ln\eta] \\ + \frac{DW_c}{D^2W_c}\left(\frac{n_1}{W_0} + n_2DW_0\right) + O(\mu^{\frac{1}{2}}\ln\mu). \end{aligned} \tag{22}$$

Thus
$$\left.\begin{aligned}\epsilon_0^{**} &= \mu^{-\frac{2}{3}}, \quad v_0^{**} = i(DW_c/D^2W_c)\,(iW_0)^{\frac{2}{3}}\, n_2,\\ \epsilon_1^{**} &= \mu^{-\frac{1}{3}}\ln\mu, \quad v_1^{**} = in_2(iW_0)^{\frac{2}{3}}\,\eta_2,\\ \epsilon_2^{**} &= \mu^{-\frac{1}{3}}, \quad \epsilon_3^{**} = 1, \quad v_3^{**} = (DW_c/D^2W_c)\,(n_1/W_0 + n_2 DW_0),\end{aligned}\right\} \tag{23}$$

and the equation satisfied by v_2^{*} is

$$\frac{d^4v_2^{**}}{d\eta^4} - i\eta DW_c\frac{d^2v_2^{**}}{d\eta^2} = \frac{i}{2}\eta^2D^2W_c\frac{d^2v_0^{**}}{d\eta^2} - iD^2W_cv_0^{**}$$
$$= (iW_0)^{\frac{2}{3}}DW_cn_2;$$

hence
$$v_2^{**} = A_1 + B_1\eta + i(iW_0)^{\frac{2}{3}}\,(iDW_c)^{\frac{1}{3}}\,n_2(v_2^{**})_{\text{part}}, \tag{24}$$

where
$$(v_2^{**})_{\text{part}} = \pi\int_0^\eta\int_0^\eta G[(iDW_c)^{\frac{1}{3}}\,\eta]\,d\eta\,d\eta$$
$$= -\eta(iDW_c)^{-\frac{1}{3}} + \eta\int_0^\eta G[(iDW_c)^{\frac{1}{3}}\,\eta]\,d\eta + (iDW_c)^{\frac{1}{3}}\left[\left(\frac{dG}{d\eta}\right)_{\eta=0} - \frac{dG}{d\eta}\right] \tag{25}$$

and
$$G(x) = \frac{i}{\pi}\int_0^\infty \exp\left[-i(xs + 1/3s^3)\right]ds$$

is that particular integral which does not grow exponentially as $x \to \pm\infty$.

The process of finding higher approximations can be continued further in a simple manner. The equations will all be of the form

$$\frac{d^4v_n^{**}}{d\eta^4} - i\eta DW_c\frac{d^2v_n^{**}}{d\eta^2} = \sum_{m=-2}^{M} L_m\frac{d^mG}{d\eta^m}, \tag{26}$$

where the L_m are polynomials in η and $n \geqq 4$. The solution of equation (26) is obtainable by repeated use of the recursion relation

$$\left(\frac{d^2}{dx^2} - x\right)\frac{d^nG}{dx^n} = \begin{cases} -1/\pi & (n = 0),\\ n\dfrac{d^{n-1}G}{dx^{n-1}} & (n \geqq 1).\end{cases} \tag{27}$$

For $-\pi < \arg i^{\frac{1}{3}}\eta = \arg x < \frac{1}{3}\pi$,

$$\int_0^x dx\int_0^x dx\,G(x) = -\frac{x}{\pi} + x\int_0^x dx\,G(x) + G'(0) - G'(x)$$
$$\sim x/\pi\left[\ln x + \tfrac{1}{3}(\ln 3 + 2\gamma - 3 + i\pi) + \frac{\pi 3^{-\frac{5}{6}}}{x\Gamma(\frac{1}{3})}(1 - i\sqrt{3})\right.$$
$$\left. + \frac{1}{x^3}\,{}_3F_0\left(1, \tfrac{2}{3}, \tfrac{4}{3}; \frac{9}{x^3}\right) + \frac{2}{3x^3}\,{}_4F_1\left(1, 1, \tfrac{4}{3}, \tfrac{5}{3}; 2; \frac{9}{x^3}\right)\right]; \tag{28}$$

(the asymptotic expansions are from Luke (1962), chapter 6, where in his notation $G(x) = Gi(x) + iAi(x)$); and the asymptotic expansions of

$$\frac{d^m}{dx^m}\int_0^x dx\int_0^x dx\,G(x),$$

and hence the asymptotic expansions of v_n^{**}, $n \geqq 4$, can then be found by term-by-term differentiation of equation (28).

If, however, only the characteristic equation is desired, it is seen that the determination of v^{**} is not necessary. Since $\bar{v}$ and $\bar{\bar{v}}$ satisfy the same equations, v^{**} only serves to bridge the gap between their various regions of validity. As can be seen from the forms of the equations which the v_n^{**} must satisfy and from equation (28), all of the v^{**} will have asymptotic forms which are dominated by polynomials in η plus $\ln\eta$ times polynomials in η. Thus the crossing of the second inner region is determined by which branch of the logarithm is to be selected, which was just seen to be $-\pi < \arg i^{\frac{1}{3}}\eta < \frac{1}{3}\pi$, and so the expansion used for $\bar{v}$ can be used for $\bar{\bar{v}}$ also under this restriction. The relations given for the v^{**} determine the asymptotic form of v in this region, but are otherwise unnecessary.

The characteristic equations corresponding to the boundaries considered in case I are then found from

$$\bar{\bar{v}} \sim -\frac{1}{\epsilon}(iW_0^3)^{\frac{1}{2}}\,\Phi_2(y,0)+\frac{1}{W_0}\,\Phi_1(y,0)-DW_0\Phi_2(y,0)+O(\epsilon). \tag{29}$$

Many of the details of calculation are similar to those of case I and need not be repeated here. The characteristic equations are, for the various cases:

Case (II–1)

The results of (I, 1) can be used exactly by replacing y by $1-y$ and c by $1-c$. It is now $1-c$ which is small.

(*Case* (II–2a)

Application of the boundary conditions to equation (29) gives

$$\frac{\Phi_1(1,0)}{\Phi_2(1,0)} \approx (iW_0^5\alpha R)^{\frac{1}{2}}+W_0\,DW_0.$$

Case (II–2b)

The boundary conditions and equation (29) give

$$\frac{d\Phi_1(1,0)}{dy}\bigg/\frac{d\Phi_2(1,0)}{dy} \approx (iW_0^5\alpha R)^{\frac{1}{2}}+W_0DW_0.$$

Case (II–3)

Again using equation (29),

$$\left[\frac{d\Phi_1(1,0)}{dy}+\alpha\Phi_1(1,0)\right]\bigg/\left[\frac{d\Phi_2(1,0)}{dy}+\alpha\Phi_2(1,0)\right] \approx (iW_0^5\alpha R)^{\frac{1}{2}}+W_0DW_0.$$

4. Case III. Critical point near a non-rigid boundary or near an extremum value of W

The solution in the first inner and first outer layer for this case are exactly as given for case II. The difference is now in matching in the second inner region, the technique being the same as in the previous cases.

In cases I and II, DW_c was assumed to be of order 1. If y_c occurs near an extremum of W, however, DW_c will also be small, and the thickness of the layer can no longer be of order $(\alpha R)^{-\frac{1}{3}}$. A layer of thickness $(\alpha R)^{-\frac{1}{4}}$ might at first glance appear to be suitable from equation (1) but, since it is incapable of producing the proper singularity, it cannot be joined to $\bar{v}$. The layer thus must be of order $(\alpha R)^{-\frac{1}{4}}$; the lowest order approximations are then solutions of

$$\frac{d^4v^{**}}{d\theta^4} - i\,[\theta(\alpha R)^{\frac{1}{4}}\,DW_c + \tfrac{1}{2}\theta^2 D^2W_c]\,\frac{d^2v^{**}}{d\theta^2} + iD^2W_c v^{**} = 0,$$

where $\theta = (y-y_c)\,(\alpha R)^{\frac{1}{4}}$. The solutions are χ_5, χ_6, χ_7, and

$$\theta(\theta + 2(\alpha R)^{\frac{1}{4}}\,DW_c/D^2W_c),$$

where, from Laplace's method,

$$\chi_5, \chi_6 = \int_{C_5,\,C_6} P^{-\frac{1}{2}}\,{}_1F_1(\tfrac{5}{4}+k, \tfrac{5}{2}; P)\,f\,dP,$$

$$\chi_7 = \int_{C7} P^{-2}\,{}_1F_1(-\tfrac{1}{4}+k, -\tfrac{1}{2}; P)\,f\,dP,$$

$$k = i^{\frac{3}{2}}(\alpha R)^{\frac{1}{2}}\,(DW_c)^2/16(D^2W_c)^{\frac{3}{2}},$$

$$f = \exp\,[(\tfrac{1}{2}iD^2W_c)^{\frac{1}{4}}\,(\theta + (\alpha R)^{\frac{1}{4}}\,DW_c/D^2W_c)\,P^{\frac{1}{2}} - \tfrac{1}{2}P]$$

and C_5 goes from 0 to i^∞, C_6 goes from 0 to $-i^\infty$, and C_7 goes from $-i^\infty$ to i^∞, passing to the right of $P = 0$.

These solutions approach the complexity of the full solutions of the Orr–Sommerfeld equation, which can be found for a parabolic profile in a similar manner. In determining the characteristic equation for this case, it would be simpler to place this boundary condition at $y = 0$ and commence from there, eliminating the need for higher approximations in this region. Owing to the added complexity in this region, the characteristic equation was not carried out for this case.

It is clear that this region cannot occur in a part where W becomes constant. However, in the case of a free surface with boundary conditions of zero shear stress and zero normal stress at the free surface, giving

$$[D^2v - (\alpha^2 + D^2\,W/W)v]_{y=1} = 0$$

and

$$[(i/\alpha R)\,D^3v - WDv + vDW]_{y=1} = 0,$$

respectively, such a layer can be expected to occur if DW vanishes at $y = 1$.

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